\( \forall n \in \mathbb{Z}, n((n^2-1)(n+2)) \text{ is divisible by } 4 \)

**Proof:**

\( \forall n \in \mathbb{Z}, n((n^2-1)(n+2)) \equiv 0 \pmod{4} \)

By CRT, there are 4 \( \pmod{4} \) cases.

1. \( \exists k \text{ s.t. } n = 4k \)
   \[ n(n^2-1)(n+2) = 4k(16k^2-1)(4k+2) \]
   \[ \therefore 4 \mid n(n^2-1)(n+2) \]
2. \( \exists k \text{ s.t. } n = 4k+1 \)
   \[ n(n^2-1)(n+2) = (4k+1)(16k^2+4k+1-1)(4k+2) \]
   \[ \therefore 4 \mid n(n^2-1)(n+2) \]
3. \( \exists k \text{ s.t. } n = 4k+2 \)
   \[ n(n^2-1)(n+2) \equiv 1 \cdot (1-1)(1+2) \equiv 0 \pmod{4} \]
4. \( \exists k \text{ s.t. } n = 4k+3 \)
   \[ n(n^2-1)(n+2) \equiv 3 \cdot (9-1)(3+2) \equiv 0 \pmod{4} \]

**Proof:** 4 cases by CRT

1. \( n \equiv 0 \pmod{4} \). In this case
   \[ n(n^2-1)(n+2) \equiv 0 \pmod{4} \]

Contrapositive: \( \forall a, b \in \mathbb{Q}, r \notin \mathbb{Q}, b \neq 0 \rightarrow a + br \notin \mathbb{Q} \)

Negation: \( \exists a, b \in \mathbb{Q}, r \notin \mathbb{Q}, a + br \in \mathbb{Q}, b \neq 0 \)

Proof by contradiction:

Assume the negation is true, find \( a, b, r \) s.t.

\[ a, b, a + br \in \mathbb{Q}, r \notin \mathbb{Q}, b \neq 0 \]

Then

\[ r = \frac{(a+br) - a}{b} \]

which is rational since \( \text{diff of } 2 \text{ rationals is rational} \)

This contradicts \( r \) is irrational

\[ \therefore \text{the negation is false} \]

\[ \therefore \text{original statement is true} \]
Then: $\sqrt{5}$ is irrational.

Proof: Assume $\sqrt{5}$ is rational.

That means $\exists p, q \in \mathbb{Z}$ s.t. $q > 0$ and $\sqrt{5} = \frac{p}{q}$

and $\gcd(p, q) = 1$

$\sqrt{5} = \frac{p}{q}$

So $5q^2 = p^2$, in other words $5 \mid p^2$.

$\frac{1}{6} = \frac{p^2}{5q^2} \Rightarrow 5 \mid p^2$ (by lemma on next page)

So $\exists k \in \mathbb{Z}$ such that $p = 5k$.

$p^2 = \frac{1}{6} \cdot 5k^2$

Then $\frac{p^2}{5q^2} = \frac{5k^2}{5q^2}$ on other words $5 \mid q^2$.

$\therefore 5 \mid q$

$\frac{1}{6} \mid p^2$

$\therefore \gcd(p, q) \neq 5$ contradicts $\gcd(p, q) = 1$

$\therefore \sqrt{5}$ is irrational.

Lemma: $\forall n \in \mathbb{Z}, 5 \mid n^2 \Rightarrow 5 \mid n$ $\equiv \forall n \in \mathbb{Z}, 5 \mid n \Rightarrow 5 \mid n^2$

Assume $n \in \mathbb{Z}$ and $5 \mid n^2$.

Proof: By CRT there are 5 cases:

$n \equiv 0, 1, 2, 3, 4 \pmod{5}$

but 1st case ($n \equiv 0$ case) is eliminated by $5 \mid n^2$.

4 cases remain:

1) $n \equiv 1 \pmod{5}$

Then $n^2 \equiv 1^2 \equiv 1 \not\equiv 0 \pmod{5}$. 

2) $n \equiv 2 \pmod{5}$

Then $n^2 \equiv 2^2 \equiv 4 \not\equiv 0 \pmod{5}$

3) $n \equiv 3 \pmod{5}$

Then $n^2 \equiv 3^2 \equiv 4 \not\equiv 0 \pmod{5}$

4) $n \equiv 4 \pmod{5}$

Then $n^2 \equiv 16 \equiv 1 \not\equiv 0 \pmod{5}$

$\therefore n^2 \not\equiv 0 \pmod{5}$

$\forall n \in \mathbb{Z}, 5 \mid n \Rightarrow 5 \mid n^2$
For $a, b \in \mathbb{Z}^+$, $g, r \in \mathbb{Z}$, $a = bg + r \Rightarrow gcd(a, b) = gcd(b, r) \wedge 0 \leq r < b$

Example: $gcd(450, 240) = gcd(240, 210)$ since $450 = 240 \cdot 1 + 210$

$= gcd(210, 30)$ since $240 = 210 \cdot 1 + 30$

$= gcd(30, 0)$ since $210 = 30 \cdot 7 + 0$

$= 30$ since $gcd(a, 0) = a$ for any $a \in \mathbb{Z}^+$

Proof: Assume $a, b \in \mathbb{Z}^+$, $g, r \in \mathbb{Z}$ and $a = bg + r$ and $0 \leq r < b$.

We show 1) $gcd(a, b) \leq gcd(b, r)$

Let $c \mid a$ and $c \mid b$ (will show $c$ also a c.d. of $b$ and $r$)

Since $r = a - bg$ and $c \mid a$ and $c \mid bg$,

$i.e. c \mid (b, r)$

2) $gcd(b, r) \leq gcd(a, b)$

similar