Let \( f : X \to Y \) and \( g : Y \to Z \) where \( Y \triangleq Y \) then the composition of \( f \) with \( g \), \( g \circ f : X \to Z \) is defined by \( (g \circ f)(x) = g(f(x)) \).

Exist \( f : \mathbb{R} \to \mathbb{R} \) be \( f(x) = 6x + 2 \)
Exist \( g : \mathbb{R} \to \mathbb{R} \) be \( g(x) = x^2 \)

then \( (g \circ f)(x) = g(f(x)) \)
\[ = g(6x + 2) \]
\[ = (6x + 2)^2 \]

\( (f \circ g)(x) = f(g(x)) \)
\[ = f(x^2) \]
\[ = 6x^2 + 2 \]

(onto)

Let \( f : X \to Y \), \( g : Y \to Z \) both be one-to-one.

Then \( g \circ f \) is also one-to-one.

Proof: Suppose \( f, g \) are as above and that
\[ (g \circ f)(x_1) = (g \circ f)(x_2) \]
\[ g(f(x_1)) = g(f(x_2)) \] (def of \( o \))
\[ f(y_1) = f(y_2) \] (\( g \) is 1-1)
\[
\therefore \ x_1 = x_2 \] (since \( f \) is 1-1)
Let \( f: X \rightarrow Y \). Then inverse image of \( y \) is

\[
\begin{align*}
f(x) &= x^2 \\
f^{-1}(1) &= \{-1, 1\} \\
f^{-1}(4) &= \{2, -2\} \\
f^{-1}(-3) &= \emptyset \\
f^{-1}(0) &= \{0\}
\end{align*}
\]

\[g(x) = 6x + 2\]

\[
\begin{align*}
g^{-1}(8) &= \{1\} \\
g^{-1}(6) &= \{10\} \\
all\ these\ images\ have\ exactly\ 1\ preimage\ since\ every\ \( f \)\ is\ a\ bijection
\end{align*}
\]

If \( g: X \rightarrow Y \) is a bijection then its inverse \( g^{-1}: Y \rightarrow X \) is defined by

\[
g^{-1}(y) = \text{the } x \text{ such that } g(x) = y
\]

\[
g^{-1}(y) = x \text{ means same as } g(x) = y
\]

\[
g^{-1}(y) = \text{the } x \text{ such that } 6x + 2 = y
\]

\[
6x = y - 2 \\
x = \frac{y - 2}{6}
\]

If \( f: X \rightarrow Y \) is a bijection then \( f^{-1} \) is too.
Pigeonhole principle: If \( f: X \to Y \) and \( X, Y \) are finite w/ \( N(X) > N(Y) \) then \( f \) is not 1-1

\[
\begin{array}{c}
\text{not 1-1} \\
\end{array}
\]

Proof uses fact that inverse images partition \( X \)

Let \( Y = \{y_1, \ldots, y_m\} \)

\( f^{-1}(y_1), f^{-1}(y_2), \ldots, f^{-1}(y_m) \) partition \( X \)

1) pairwise disjoint

2) union is equal \( X \)

so \( N(X) = N(f^{-1}(y_1)) + \ldots + N(f^{-1}(y_m)) \)

is \( f \) 1-1 all these are are 1's

sum too small

If \( X, Y \) are finite sets w/ \( N(X) = N(Y) \) and \( f: X \to Y \)
then \( f \) is 1-1 \( \iff \) \( f \) is onto

For finite sets \( X, Y \), there is a 1-1, onto \( f: X \to Y \iff N(X) = N(Y) \)

Def: \( A \) has same cardinality as \( B \) iff \( \exists f: A \to B \) that is a bijection.

For any sets \( A, B, C \)

Properties: \( A \) has same card as \( A \) \( \iff \) \( i_A: A \to A \) is a bijection

\( A \) same card \( B \to B \) has card as \( A \) \( \iff \) \( f^{-1} \)

\( A \) same card \( B \land B \) same card \( C \to A \) same card \( C \) \( \iff \) \( g \circ f \)

Def: \( A \) is countable iff \( A \) is finite or \( A \) is countably infinite

\( A \) is uncountable iff \( A \) not countable

\( A \) same card as \( \mathbb{Z}^+ \)
\{ 2, 4, 6, 8, \ldots \} \text{ is countably inf : let } f(x) = \frac{x}{2} \\
\text{f is 1-1, onto}

\{ 1, 2, 3, \ldots \}

\mathbb{Z} \text{ is countably inf : } \{ 1, 2, 3, 4, 5, \ldots \}

\{ \ldots, -2, -1, 0, 1, 2, \ldots \}

f(x) = \begin{cases} 
\frac{x}{2} & \text{if } x \text{ even} \\
-\frac{x-1}{2} & \text{if } x \text{ odd}
\end{cases}

f \text{ 1-1, onto}

(0, 1) = \{ x \in \mathbb{R} \mid 0 < x < 1 \} \text{ is uncountable.}

\text{Proof: Any function } f: \mathbb{Z}^+ \to (0, 1) \text{ is either not 1-1 or not onto.}

\text{Let } f: \mathbb{Z}^+ \to (0, 1). \text{ [want f not onto -- need } y \in (0, 1) \text{ s.t. no } x \in \mathbb{Z}^+ \text{ sats } f(x) = y]\)

f(1) = 0. a_1 a_2 a_3 \ldots \text{ digits in decimal rep}

f(2) = 0. a_{12} a_{22} a_{32} \ldots \text{ (use term rep)}

f(3) = 0. a_{31} a_{32} a_{33} \ldots \text{ (use term rep)}

\text{let } y \neq f(1) \text{ by making them differ in } 1^{\text{st}} \text{ digit } \gamma \neq f(2) \text{ 1st digit}

\text{2nd digit}
Let $i$th digit of $γ = \sum_{i \in \mathbb{Z}} 1$ if $i$th digit of $f(i) \neq 1$.