Problem 0: Review old quizzes, homeworks, and exams.

Problem 1: Write negations of the following sentences.

(a) Rafael Palmerio uses steroids but he is not a good hitter.
(b) If Burt drives a Camaro then Candy likes Burt.
(c) Anyone who drives a BMW is rich.
(d) All dairy farmers have a machine that can milk any cow.

Problem 2: Show that the following argument form is valid.

\[
\begin{align*}
& p \to q \\
& \neg p \to r \\
\therefore q \lor r
\end{align*}
\]

Problem 3: Design a circuit with four inputs that outputs 1 if and only if at least three of its inputs are 1.

Problem 4: Convert the decimal numbers in the following list to 8-bit two’s complement and convert the binary numbers to decimal. When converting binary numbers, interpret the number first as an unsigned number and then as a signed number encoded using 8-bit two’s complement.

- 100_{10}
- −53_{10}
- 01001110_2
- 10101010_2

Problem 5: Let \( L(x, y) \) be the predicate “\( x \) lives in \( y \)”. Let \( S(x, y) \) be the predicate “\( x \) shops at \( y \)”. Let the domains of the variables and the truth of the predicates for particular values be as indicated by the tables given below. Use \( P \) for the set of people, \( C \) for the set of cities (ignoring the fact that Columbia and Timonium are not cities), and \( S \) for the set of stores.
Write each of the following English statements symbolically and determine whether they are true or false. Explain your answers briefly.

(a) Eastman shops at Giant.

(b) Everyone who lives in Columbia shops at Giant.

(c) No one shops at two different stores.

(d) There is a store whose patrons all live in Columbia.

(e) There is a place whose residents all shop at the same store.

Problem 6: Prove or disprove: for any integers $a, b, c,$ and $d$, if $a + b | d$ and $a + c | d$ then $b + c | d$.

Problem 7: Find an integer $r$ such that $0 \leq r < 13$ and $11^8 \equiv r \pmod{13}$.

Problem 8: Prove that $\sqrt{\frac{5}{2}}$ is irrational.

Problem 9: Rewrite each of the following using summation notation.

(a) $4 + 7 + \cdots + 31$

(b) $8 + \cdots + n!n^2$

(c) $-2 + 8 - \cdots + 8n^2$

Problem 10: Find a formula for $\sum_{i=1}^{n} 4i + 3$. Check your answer by proving it by induction.

Problem 11: Find the smallest $k$ such that any amount of at least $k$ cents can be made with 4-cent and 5-cent stamps. Prove your answer.
Problem 12: Prove that the following code fragment correctly computes $\lfloor \log_2 a \rfloor$.

[Precondition: $A \geq 1.0, l = 0, b = A$]

```plaintext
while (b >= 2.0)
  b = b / 2.0  // note that A, b are reals, not integers
  l = l + 1
end while

[Postcondition: $l = \lfloor \log_2 a \rfloor$.]
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(Hint: part of your invariant can be $b \cdot 2^l = a$; review properties of floors and ceilings.)

Problem 13: Prove that for any sets $A$ and $B$, $A \cup (A \cap B) = A$ using no properties of sets other than the definitions of the set operations.

Problem 14: Prove that, for any sets $A$, $B$, $C$, and $D$, if $C \subseteq A - B$ and $D \subseteq B - A$ then $C$ and $D$ are disjoint.

Problem 15: Prove or disprove: for any sets $A$ and $B$, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

Problem 16: A state issues license plates in two formats: either two letters followed by three numbers, or 5 numbers with a letter somewhere between them (but not at the ends). How many different license plates can the state issue?

Problem 17: Imagine a game like poker but played with 4 card hands. Which should be a better hand: one pair or “melting pot”, which is four cards all of different ranks and different suits.

Problem 18: 50 people were surveyed about what TV shows they watch. 21 reported that they watch The Amazing Race, 25 watch Veronica Mars, and 11 watch Arrested Development. 3 watch AD and TAR, 9 watch TAR and VM, 3 watch VM and AD and 1 watches all 3. Fill in the Venn diagram that shows how many people watch each possible combination of shows.

Problem 19: 12 people are to be seated for a jury. The jury pool consists of 20 people: 7 college students, 10 retirees, and 3 working professionals. How many possible juries are there? How many have an equal number from each group? How many include no retirees? How many include at most 3 retirees? How many include all the college students? How many include more working professionals than college students?

Problem 20: The game Can’t Stop is played with 4 indistinguishable 6-sided dice. How many distinct outcomes are there of rolling the dice?

Problem 21: Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by $f(x) = (x - 1)(x + 2)(x + 3)$. Is $f$ 1-1? Is $f$ onto? Explain your answers.
Problem 22: Prove that the set of squares of integers \( \{0, 1, 4, ...\} \) is countably infinite by finding a bijection between that set and \( \mathbb{Z}^+ \).

Problem 23: Define \( f : \mathbb{R}^{\text{nonneg}} \rightarrow \mathbb{R} \) by \( f(x) = x^2 + 3x + 7 \) and \( g : \mathbb{R}^{\text{nonneg}} \rightarrow \mathbb{R} \) by \( g(x) = 3x^2 - 9x + 100 \). Prove that \( f(x) \) is \( \Theta(g(x)) \).