Thm $\forall a \in \mathbb{Q}, b \in \overline{\mathbb{Q}}, \quad a+b \in \overline{\mathbb{Q}}$

Proof: Assume thm is false: $\exists a \in \mathbb{Q}, b \in \overline{\mathbb{Q}}$ s.t. $a+b \notin \overline{\mathbb{Q}}$

$a+b \in \mathbb{Q}$

Find such an a, b.

Then $b = (a+b) - a$ where $a+b \in \mathbb{Q}$ and $a \in \mathbb{Q}$

$\therefore b \in \mathbb{Q}$ by closure of $\mathbb{Q}$ under $-$

$\Rightarrow\Leftarrow$

$\therefore$ Thm is true

$\forall n \in \mathbb{Z}$, if n is not even then n^2 is not even $\quad n^2 > 9$

$\equiv \forall n \in \mathbb{Z}$ if n odd then n^2 odd by GRT (Corr) $\quad |n| > 3$

<u>Thm:</u> $\forall n \in \mathbb{Z}$, if n^2 is even then n is even

$\qquad\qquad 2 \mid n^2 \qquad\qquad 2 \mid n$

$\cancel{a \equiv b \pmod{m}}$

$\cancel{\sqrt{a} \equiv \sqrt{b} \pmod{m}}$ $\qquad n^2 \equiv 0 \pmod{2} \qquad n \equiv 0 \pmod{2}$

Proof: We prove contrapositive: $\forall n \in \mathbb{Z}$, n is odd $\rightarrow n^2$ is odd

$n \equiv 1 \pmod{2} \rightarrow n^2 \equiv 1 \pmod{n}$

Assume $n \in \mathbb{Z}$ and n is odd.

$n \equiv 1$, so $n^2 \equiv 1^2 \equiv 1 \pmod{n}$

Then $\exists k \in \mathbb{Z}$ s.t. $n = 2k+1$

so $n^2 = (2k+1)^2$

$= (4k^2 + 4k) + 1$

$= 2(2k^2 + 2k) + 1$

$\therefore \exists \ell \in \mathbb{Z}$ s.t. $n^2 = 2\ell + 1$

namely $2k^2 + 2k$

$\therefore n^2$ is odd

Thm: $\forall n \in \mathbb{Z},\ 3 \mid n^2 \rightarrow 3 \mid n$

(or $\forall n \in \mathbb{Z}\ \ n^2 \equiv 0 \pmod 3 \rightarrow n \equiv 0 \pmod 3$)

Proof: We prove contrapositive: $\forall n \in \mathbb{Z},\ n \not\equiv 0 \pmod 3 \rightarrow n^2 \not\equiv 0 \pmod 3$

Assume $n \in \mathbb{Z}$ and $n \not\equiv 0 \pmod 3$

By QRT $n \equiv 0 \pmod 3 \vee n \equiv 1 \pmod 3 \vee n \equiv 2 \pmod 3$

By Elim $n \equiv 1 \pmod 3 \vee n \equiv 2 \pmod 3$

Case 1 ($n \equiv 1 \pmod 3$): $n^2 \equiv 1^2 \equiv 1 \not\equiv 0 \pmod 3$

Case 2 ($n \equiv 2 \pmod 3$): $n^2 \equiv 2^2 \equiv 4 \equiv 1 \not\equiv 0 \pmod 3$

$\therefore$ (by cases) $n^2 \not\equiv 0 \pmod 3$

Thm : $\sqrt{2} \notin \mathbb{Q}$.

Well Ordering principle: any non-empty set of positive ints has a least elt

Proof: Assume $\sqrt{2} \in \mathbb{Q}$

Then $\exists\, a, b \in \mathbb{Z}$ s.t. $b \neq 0$ $\frac{a}{b} = \sqrt{2}$ where a, b have no common factors

Then $\frac{a^2}{b^2} = 2$

$a^2 = 2b^2$

so $2 \mid a^2$ and a^2 is even

$\therefore$ by prev Thm , a is even

$\therefore$ $a = 2k$ for some $k \in \mathbb{Z}$

Now $(2k)^2 = 2b^2$

so $\cancel{4}^{2} k^2 = \cancel{2} b^2$

$2k^2 = b^2$

$\therefore$ $2 \mid b^2$, so b^2 is even and hence b is even,

so $\exists\, \ell \in \mathbb{Z}$ s.t. $b = 2\ell$

$\Rightarrow\Leftarrow$ (a, b have a common factor of 2)

$\therefore \sqrt{2} \notin \mathbb{Q}$