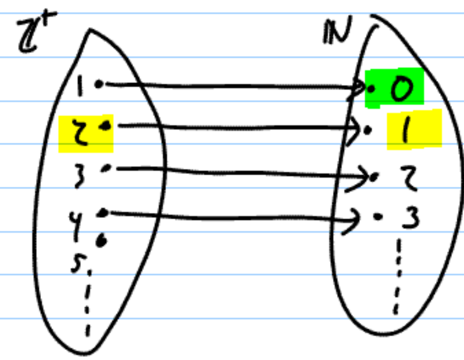
Thm: $\mathbb{Z}^+$ has the same cardinality as $\mathbb{N}$.

Proof: Define $f : \mathbb{Z}^+ \to \mathbb{N}$ by $f(n) = n - 1$



Then f is 1-1: Suppose $f(n_1) = f(n_2)$
then $n_1 - 1 = n_2 - 1$
and $n_1 = n_2$

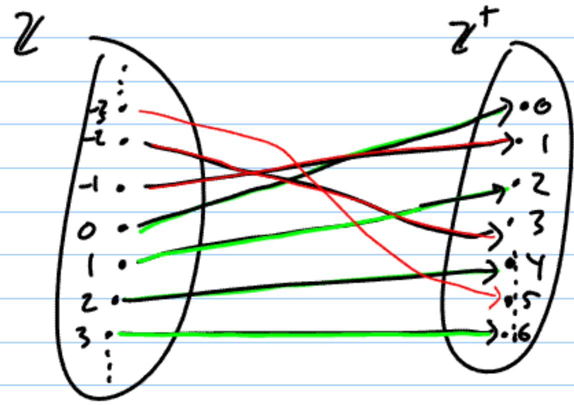
Then f is onto: Assume $n \in \mathbb{N}$. Let $m = n+1$
then $f(m) = n+1-1 = n$
$\therefore \exists m \in \mathbb{Z}^+$ s.t. $f(m) = n$

$\exists$ a bijection from $\mathbb{Z}^+ \to \mathbb{N}$

$\therefore \mathbb{Z}^+$ has same cardinality as $\mathbb{N}$

Thm: $\mathbb{Z}$ has the same cardinality as $\mathbb{Z}^+$.

Proof: Define $f : \mathbb{Z} \to \mathbb{Z}^+$ by $f(n) = \begin{cases} 2n & \text{if } n \geq 0 \\ -2n-1 & \text{if } n < 0 \end{cases}$



f is 1-1

f is onto

$\therefore \exists$ bijection $f : \mathbb{Z} \to \mathbb{Z}^+$

$\therefore \mathbb{Z}$ has same card as $\mathbb{Z}^+$

Is there a bijection $f : \mathbb{Z}^+ \to \mathbb{Z}$? Yes (use inverse of ↑)

What is a function $f : \mathbb{Z}^+ \to X$ A list of elts of x!

$\mathbb{Z} = \{ 0, 1, -1, 2, -2, 3, -3, 4, -4, \ldots \}$

$f(1)$ $f(2)$ $f(3)$ ……

$f(17) = f(34)$?

If f is 1-1 and onto then the corresponding list 1) no duplicates 2) is complete

Let $\Sigma = \{a, b\}$ Can U list all elts of Σ^*

set of strings using chars in alphabet Σ

"", a, b, ab, ba, aa, bb, aab, aba, baa, bbb, bba, bab, abb, aaa, -----..

", a, b, aa, ab, ba, bb, aaa, aab, aba, abb, baa, bab, bba, bbb, aaaa, ----

```
public String foo(int index)
{
   // returns string at loc index in this list
}
```

abbbaaba

we can write this fxn ; it will be 1-1 and onto

$\therefore \exists$ bijection from $\mathbb{Z}^+ \rightarrow \Sigma^*$

$\therefore \Sigma^*$ is countably infinite

$\mathbb{N} \times \mathbb{N}$

(0, 0)	(0, 1)	(0, 2)	(0, 3)	- - - -
(1, 0)	(1, 1)	(1, 2)	(1, 3)	- - - -
(2, 0)	(2, 1)	(2, 2)	(2, 3)	- - -
(3, 0)	(3, 1)	- - - - -		
⋮				

→ (0, 0), (0, 1), (0, 2), (0, 3), - - - - - - - -

(0, 0), (0, 1), (1, 0), (0, 2), (1, 1), (2, 0), - - - - - -

given (m, n) we can compute index on list (onto)
also, no duplicates on the list (1-1)

$\mathbb{N} \times \mathbb{N}$ has same cardinality as $\mathbb{Z}^+$

$A = \{ f \mid f \text{ is a function from } \mathbb{Z} \text{ to } \mathbb{Z} \}$

Thm: A is uncountable

Proof: [need to show there is no bijection from $\mathbb{Z}^+$ to A]

$\forall f : \mathbb{Z}^+ \to A$, f is not 1-1 or not onto

Suppose $f : \mathbb{Z}^+ \to A$ [want: f is not 1-1 or f is not onto]

$\exists g \in A$ s.t. $\forall n \in \mathbb{Z}^+\ f(n) \neq g$

$f(x) = x$

$g(x) = x+1$

$f = g$? No

$f(1) \neq g(1)$

Construct $g : \mathbb{Z} \to \mathbb{Z}$ by letting $g(x) = \begin{cases} 0 & \text{if } n \leq 0 \\ f(n)(n)+1 & \text{if } n > 0 \end{cases}$

Need $g \neq f(1)$ — make them differ at 1

$g \neq f(2)$ → let $g(1) = f(1)(1)+1$

$g \neq f(3)$ → let $g(2) = f(2)(2)+1$

$\vdots$

$g \in A$, $g \neq f(n)$ for all $n \in \mathbb{Z}^+$

list of all fcns from $\mathbb{Z} \to \mathbb{Z}$:

item 1 on list) the zero function ($f(n)=0\ \forall n$)

2) 45° line ($f(n)=n$)

3) ⋮ ($f(n)=n^2$)

we can find a fcn not on the list

$\therefore$ list is complete (no fcn $\mathbb{Z}^+ \to A$ is onto)

$\therefore$ no bijection exists

$\therefore$ $\mathbb{Z}^+$, A have different cardinalities

$\therefore$ A uncountable