Constructive Induction or Substitution – a method for solving recurrences when you have guessed the general form but don’t know the constants.

We know that $\sum_{i=1}^{n} i = \frac{1}{2}n^2 - \frac{1}{2}n$. What is $\sum_{i=1}^{n} i^2$? Since the formula for the first sum was a quadratic, we might guess that the formula for the second sum is a cubic, that is, it is of the form $an^3 + bn^2 + cn + d$.

We can use the steps we would go through to prove our formula by induction to tell us what $a$, $b$, $c$, and $d$ have to be in order to make the proof work.

Our induction step would start with the assumptions $\sum_{i=1}^{n-1} i^2 = a(n-1)^3 + b(n-1)^2 + c(n-1) + d$ and $n > 0$. We would have to prove that $\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$. In order for this to work, we need

$$\sum_{i=1}^{n} i^2 = an^3 + bn^2 + cn + d$$

[splitting sum]

$$a(n-1)^3 + b(n-1)^2 + c(n-1) + d + n^2 = an^3 + bn^2 + cn + d$$

[ind. hyp.]

$$an^3 + (b - 3a + 1)n^2 + (3a - 2b + c)n + (d - a + b - c) = an^3 + bn^2 + cn + d$$

The only way two polynomials can be equal for all values of $n$ is if they are in fact the same polynomial – all the coefficients must be the same. That leads to the system of equations

$$b - 3a + 1 = b$$

$$3a - 2b + c = c$$

$$d - a + b - c = d$$

Which means that $a = \frac{1}{3}$, $b = \frac{1}{2}$, and $c = \frac{1}{6}$. We can also get $d = 0$ (where? hint – what part of an induction proof haven’t we used yet?) which means that

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n+1)(2n+1)}{6}.$$