\textbf{Quicksort:} \( \text{quicksort}(p, r) \)
\begin{align*}
\text{if } (p < r) & \quad \text{partition}(p, r) \\
& \quad \text{randomly swap } arr[r] \\
g = \text{partition}(p, r) & \quad i = p - 1 \\
g \text{ quicksort}(p, g - 1) & \quad \text{for } j = p \text{ to } r - 1 \text{ do} \\
g \text{ quicksort}(g + 1, r) & \quad \text{if } (arr[j] < \text{pivot}) \\
& \quad \text{swap } arr[j] \leftrightarrow arr[i + 1] \\
& \quad i = i + 1
\end{align*}

Let \( X = \text{sum of } \# \text{ showing on } 2 \text{ dice} \)
\begin{align*}
\text{Then } E(X) & = 2 \cdot \frac{1}{36} + 3 \cdot \frac{1}{36} + \cdots + 12 \cdot \frac{1}{36} \\
& = 7 \# \text{ on } 1\text{st die} \\
\text{let } X = X_1 + X_2 & \quad \text{on } 2\text{nd die} \\
& \leq \text{pivot}\quad \geq \text{pivot}
$E[X_i] = E[X_j] = 3^{1/2}$

$E[X] = E[X_i + X_j]$

$= E[X_i] + E[X_j]$

$= 3^{1/2} + 3^{1/2} = 7$

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Let $X$ = # comparisons done in partition at $x$

$X = X_{12} + X_{13} + \cdots + X_{1n} + X_{23} + X_{24} + \cdots + X_{n-1,n}$

# of times $i$th smallest compared to $j$th smallest

so $E[X] = E[X_{12}] + \cdots + E[X_{n-1,n}]$

$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} E[X_{ij}]$

$E[X_{ij}] = 0 \cdot P(\text{not compared})$

$+ 1 \cdot P(\text{are compared})$

$= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(\text{i$th$ smallest compared to j$th$ smallest})$
\[\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} P(\text{among } i\text{th through } j\text{th smallest, } \text{i}^{th} \text{ or } j^{th} \text{ chosen as pivot before the others})\]

\[= \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j-i+1}\]

\[= \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k}\]

\[< 2 \sum_{i=1}^{n-1} \frac{n}{k} \]

\[\int_{1}^{\ln n} \frac{1}{x} \, dx = \ln x \bigg|_{1}^{\ln n} = 2 \sum_{i=1}^{n-1} O(\ln n) = O(n \ln n)\]
Can't do a comparison-based sort in better than \( O(n \ln n) \)

look at sorts as decision trees

for i = 1 to n-1
  for j = 1 to n-i
    if arr[j] > arr[j+i] out of order
      swap them
Decision tree for bubble sort, input size 3

$n! = 3! = 6$ leaves at bottom of tree

$4 \geq 1$ for each possible ordering

binary $n$ with $h$ levels has $\leq 2^{h-1}$ leaves

we need $2^{h-1} \geq n!$

$h-1 \geq \log_2 n!$
\[ n = O(n \log n) \]

since \( \log n \) is \( O(n \log n) \)