Then: \( \{ f \mid f: \mathbb{N} \to \mathbb{N} \} \) is uncountable

Proof: “\( S \) is countable”: \( \exists f: \mathbb{N} \to S \text{ s.t. } f \text{ is bij} \)

“\( S \) is uncountable”: \( \forall f: \mathbb{N} \to S \) (\( f \) is not a bijection)

Suppose \( f: \mathbb{N} \to S \) is a bijection. Then \( f \) is onto.

[goal: contradiction (probably contradict that \( f \) is onto)]

\[ f(0) = 0, 1, 8, 27, 64, 125, 216, \ldots \]

\[ f(1) = 2, 3, 4, 5, 6, 7, 8, 9, \ldots \]

\[ f(2) = 10, 11, 12, 13, 14, 15, \ldots \]

Let \( g(0) = f(0)(0) + 1 \) now \( g \neq f(0) \) since they disagree at 0

\[ g = 1, 2, 3, \ldots \]

\[ g(1) = f(1)(1) + 1 \]
in general \( g(n) = f(n)(n)+1 \) since that makes

\[ g \neq f(n) \]

since they are \( \neq \) at \( n \)

But \( g : \mathbb{N} \rightarrow \mathbb{N} \), so if \( f \) is really onto

( if list is complete)

then \( g \) should be on list \( (g = f(k) \) for some \( k) \)

\( \Rightarrow \)

\( \therefore \) \( f \) not a bijection
Note where this fails for \( S = \{ w \mid w \text{ is a finite string over } \{a, \ldots, z\} \} \)

Suppose \( f: \mathbb{N} \to S \) is onto \( \{ f \text{ is a complete list of finite strings over } \{a, \ldots, z\} \} \)

Construct \( w \) that should be on the list but isn’t:

let \( i \text{th char of } w = \text{charAt}(i) \oplus 1 \text{ if } f(i).\text{last}() > ;
\( 'a' \) otherwise

now \( w \neq f(0) \) since they differ at index 0
\( \neq f(1) \) and so forth

so \( w \) isn’t on the list — but should it be?

NO — \( w \) is an infinite string, \( S \) includes only finite strings
So maybe \( S = \{ w \mid w \text{ is a finite string over } \{a, b, c\} \text{ and } \exists i \leq \infty \} \) is countable?

Construct a list of elements of \( S \):

\[
\begin{align*}
e & \quad e \\
a & \quad a \\
\text{aa} & \quad a \\
\text{aaa} & \quad a \\
\text{aaaa} & \quad a \\
\vdots & \quad \vdots \\
ab & \quad a \\
\text{aab} & \quad a \\
\text{aaab} & \quad a \\
\vdots & \quad \vdots \\
\text{ab} & \quad b \\
\text{aab} & \quad b \\
\text{aaab} & \quad b \\
\vdots & \quad \vdots \\
b & \quad b \\
\text{bab} & \quad b \\
\text{baab} & \quad b \\
\vdots & \quad \vdots \\
b & \quad b \\
\text{bb} & \quad b \\
\text{bba} & \quad b \\
\text{bbab} & \quad b \\
\vdots & \quad \vdots \\
\text{bb} & \quad b \\
\text{bbb} & \quad b \\
\text{bbab} & \quad b \\
\vdots & \quad \vdots \\
\end{align*}
\]

given \( w \), can you compute
index of \( w \) on list? \ NO! \ YES!

This is the list we want.
\( S \) is countably infinite.
∀n ≥ k, P(n) 

some predicate defined on \(\mathbb{N}\)

To prove: induction!

Show \(P(k)\) is T base case

Show \(∀m ≥ k, P(m) → P(m+1)\) inductive step

\[ P(0) \]
\[ ∀m ≥ 0, P(m) → P(m+1) \]
\[ P(0) → P(1) \]
\[ P(1) → P(2) \]
\[ P(2) → P(3) \]