PT for reg lang: If \( L \) is regular then \( \exists n \)

\[
\text{s.t. } \forall w \in L, |w| \geq n \text{ then } w \text{ can be written } wx_1x_2y \text{ where } x_i, y \in \Sigma^*
\]

PT for CFL: If \( L \) is CFL then \( \exists k \)

\[
\text{s.t. } \forall w \in L, |w| \geq k \text{ then } w = uvxyz
\]

where \( |v| > 0 \) or \( |y| > 0 \) and \( u, v, x, y, z \in \Sigma^* \) and \( |vxy| \leq k \)

Observation: since \( V - \Sigma \) is finite, if string \( w \) must have a really tall parse tree, then some nonterminals must have been repeated.
Suppose $G = (V, \Sigma, R, S)$

consider a parse tree of height $|V - \Sigma|$

\[ \begin{array}{c}
S \\
A \sqcup X \\
A \\
B \\
C \\
\vdots \\
D \\
a \\
b
\end{array} \]

$\phi(b) = \text{fan out of } b$

$= \text{max fan out of any rule}$

$= \text{most chars on right side of rules}$

most chars in $w$ w/ parse tree of height $|V - \Sigma| = \phi(b)^{|V - \Sigma|}$

if $w$ is longer than $\phi(b)^{|V - \Sigma|}$ then its parse tree has height $|V - \Sigma|$
If height is $\geq |V-\Sigma|$ then some nonterminal is repeated on the longest branch in parse tree that has fewest $\Delta$ of nodes

We know $|vy|>0$ otherwise is smaller parse tree.

Cut out or repeat derivation $A \Rightarrow^* A_1 y \Rightarrow^* v A y$ as many times as we want.

$A \Rightarrow^* A_1 y \Rightarrow v v A y y \Rightarrow v v v A y y y \Rightarrow v v v v A y y y y$. 

a^n b^m c^n is not CF:

Suppose it is. Then \( \exists k \in \mathbb{N}, \ l \in \mathbb{N} \geq k \Rightarrow \exists w \in \{a,b,c\}^+ \) such that \( w \in L \) and \( w = uvxyz \) where \( uv^i x y^j z \in L \).

Find that \( k \),

Let \( w = a^k b^k c^k \).

Suppose \( w = uvxyz \) where \( |vy| > 0 \)

Suppose \( v \) is all \( a \)'s

\( y \) must be \( b \)'s, \( c \)'s to balance \( a \)'s

but then repeating \( y \) makes \( b \)'s, \( c \)'s out of order.

Suppose \( v \) is all \( b \)'s then \( y \) is all \( c \)'s

and so \( uv^ixy^jz \) has too few \( a \)'s.

Suppose \( v \) is all \( c \)'s then \( y \) is all \( a \)'s

and so \( uv^ixy^jz \) has too few \( b \)'s.

Suppose \( v \) is combo of \( \geq 2 \) diff letters

then repeating \( v \) puts chars out of order.
Suppose \( v \) is \( e \). Then \( y \not= e \) and \( \uparrow \) applies to \( v \).

**CFL not closed under \( \land \)**

\[
a^mb^n c^m \land a^mb^n c^n = a^mb^n c^n
\]

\[\uparrow\]

CFL \( \Rightarrow \) non-CFL

**CFL are closed under \( \ast \)**

\( G_1 = (V_1, \Sigma_1, R_1, S_1) \) \hspace{1cm} \( G_2 = (V_2, \Sigma_2, R_2, S_2) \)

\( S \rightarrow S_1 S_2 \) (copy rules from \( R_1, R_2 \) above)

**Closure under \( \ast \)**

\[
S \rightarrow S_1 S_1 \left| e \right.
\]

**Closure under \( \lor \)**

\[
S \rightarrow S_1 \left| S_2
\right.
\]
closures under - ? NO - if so, with closure over U we'd have 1 two