Problem 0: Review old quizzes, homeworks, and exams.

Problem 1: Write negations of the following sentences.

(a) Rafael uses steroids but he is not a good hitter.

Either Rafael doesn’t use steroids or he is a good hitter.

(b) If Burt drives a Camaro then Candy likes Burt.

Burt drives a Camaro but Candy doesn’t like him.

(c) Anyone who drives a BMW is rich.

Someone who is not rich drives a BMW.

(d) All dairy farmers have a machine that can milk any cow.

Some dairy farmers's machines all fail to milk some cow.

Problem 2: Show that the following argument form is valid.

\[
\begin{align*}
p & \rightarrow q \\
\neg p & \rightarrow r \\
\therefore \quad & q \lor r
\end{align*}
\]
Problem 3: Design a circuit with four inputs that outputs 1 if and only if at least three of its inputs are 1.
**Problem 4:** Let \( L(x, y) \) be the predicate “\( x \) lives in \( y \)”. Let \( S(x, y) \) be the predicate “\( x \) shops at \( y \)”. Let the domains of the variables and the truth of the predicates for particular values be as indicated by the tables given below. Use \( P \) for the set of people, \( C \) for the set of cities (ignoring the fact that Columbia and Timonium are not cities), and \( S \) for the set of stores.

<table>
<thead>
<tr>
<th></th>
<th>Columbia</th>
<th>Baltimore</th>
<th>Timonium</th>
<th>Giant</th>
<th>Wegman’s</th>
<th>SuperFresh</th>
</tr>
</thead>
<tbody>
<tr>
<td>Eastman</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Glenn</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Lawrie</td>
<td>x</td>
<td></td>
<td></td>
<td>x</td>
<td>x</td>
<td></td>
</tr>
<tr>
<td>Binkley</td>
<td></td>
<td>x</td>
<td></td>
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<td>x</td>
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<tr>
<td>Hall</td>
<td>x</td>
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<td>x</td>
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</tbody>
</table>

Write each of the following English statements symbolically and determine whether they are true or false. Explain your answers briefly.

(a) Eastman shops at Giant. \( S(Eastman, Giant) \) – true

(b) Everyone who lives in Columbia shops at Giant. \( \forall x \in P, L(x, Columbia) \rightarrow S(x, Giant) \) – true since Eastman, Glenn, and Lawrie are the only Columbia residents and all shop at Giant.

(c) No one shops at two different stores. \( \forall x \in P, \forall s_1, s_2 \in S, S(x, s_1) \land S(x, s_2) \rightarrow s_1 = s_2 \) – false since Glenn shops at both Giant and Wegman’s.

(d) There is a store whose patrons all live in Columbia. \( \exists s \in S, \forall x \in P, S(x, s) \rightarrow L(x, Columbia) \) – true since the only patron of Wegman’s (Glenn) lives in Columbia.

(e) There is a place whose residents all shop at the same store. \( \exists c \in C, \exists s \in S, \forall p \in P, L(p, c) \rightarrow \exists (p, s) \) – true since all residents of Columbia shop at Giant.

**Problem 5:** Prove or disprove: for any integers \( a, b, c, \) and \( d \), if \( a + b \mid d \) and \( a + c \mid d \) then \( b + c \mid d \).

Counterexample: let \( a = 2, b = 0, c = 3, d = 10 \). \( 2 + 0 \mid 10 \) and \( 2 + 3 \mid 10 \) but \( 0 + 3 \nmid 10 \).

**Problem 6:** Find an integer \( r \) such that \( 0 \leq r < 13 \) and \( 11^8 \equiv r \pmod{13} \).

\[
11^8 \equiv (-2)^8 \equiv ((-2)^4)^2 \equiv 16^2 \equiv 3^2 \equiv 9 \pmod{13}
\]

**Problem 7:** Prove that \( \sqrt{\frac{1}{6}} \) is irrational.

We first claim that for all integers \( n \), if \( n^2 \equiv 0 \pmod{6} \) then \( n \equiv 0 \pmod{6} \) (prove it!). Now assume that \( \sqrt{\frac{1}{6}} \) is rational. Then by the definition of rational there are integers \( p \) and \( q \) such that \( \frac{p}{q} = \sqrt{\frac{1}{6}} \). We may assume, without loss of generality, that \( \gcd(p, q) = 1 \). But then \( 6p^2 = q^2 \) and so \( q^2 \equiv 0 \pmod{6} \). Then by our first claim, \( q \equiv 0 \pmod{6} \), and so \( q = 6k \) for some integer \( k \). By substitution we now have
\[6p^2 = (6k)^2\] and hence \(p^2 = 6k^2\), so \(p^2 \equiv 0 \pmod{6}\) and (by the first claim again), \(p \equiv 0 \pmod{6}\). Now both \(p\) and \(q\) are divisible by 6, so \(\gcd(p, q) \geq 6\), which contradicts \(\gcd(p, q) = 1\). Therefore \(\sqrt{\frac{1}{6}}\) is irrational.

**Problem 8:** Rewrite each of the following using summation notation.

(a) \(4 + 7 + \cdots + 31\)

\[\sum_{i=1}^{10} 3i + 1\]

(b) \(8 + \cdots + n!n^2\)

\[\sum_{i=2}^{n} it^2\]

(c) \(-2 + 8 - \cdots + 8n^2\)

\[\sum_{i=1}^{2n} (-1)^i \cdot 2i^2\]

**Problem 9:** Find a formula for \(\sum_{i=1}^{n} 4i + 3\).

\[
\sum_{i=1}^{n} 4i + 3 = \sum_{i=1}^{n} 4i + \sum_{i=1}^{n} 3
\]

\[= 4 \sum_{i=1}^{n} i + 3n\]

\[= 4 \cdot \frac{n(n + 1)}{2} + 3n\]

\[= 2n(n - 1) + 3n\]

\[= 2n^2 + 5n\]

**Problem 10:** Find the smallest \(k\) such that any amount of at least \(k\) cents can be made with 4-cent and 5-cent stamps. Prove your answer.

\(k = 12\) works.

Base cases \((k = 12, 13, 14, 15)\): \(12 = 3 \cdot 4, 13 = 2 \cdot 4 + 5, 14 = 4 + 2 \cdot 5, 15 = 5 \cdot 5\).

Inductive step: Let \(k \geq 16\) and suppose any amount from 12 cents to \(k - 1\) cents can be made with 4- and 5-cent stamps. Then, since \(12 \leq k - 4 \leq k - 1\) because \(k \geq 16\), the inductive hypothesis says \(k - 4\) cents can be made with 4- and 5-cent stamps. Let \(a\) and \(b\) be the number of 4- and 5-cent stamps used to make \(k - 4\) cents (so \(k - 4 = 4a + 5b\)). Now \(k\) cents can be made with \(a + 1\) 4-cent stamps and \(b\) 5-cent stamps \((k = k - 4 + 4 = 4a + 5b + 4 = 4(a + 1) + 5b)\).
Problem 11: Prove that for any sets $A$ and $B$, $A \cup (A \cap B) = A$ using no properties of sets other than the definitions of the set operations.

Let $x \in A \cup (A \cap B)$. Then either $x \in A$ or $x \in A \cap B$ by definition of $\cup$. In the former case we immediately have $x \in A$. In the latter case, $x \in A$ and $x \in B$ by definition of $\cap$. So in either case $x \in A$. Therefore $A \cup (A \cap B) \subseteq A$.

Let $x \in A$. Then $x \in A \cup (A \cap B)$, so by definition of $\cup$, $x \in A \cup (A \cap B)$. Therefore $A \subseteq A \cup (A \cap B)$.

Since $A \cup (A \cap B) \subseteq A$ and $A \subseteq A \cup (A \cap B)$, $A = A \cup (A \cap B)$.

Problem 12: Prove that, for any sets $A$, $B$, $C$, and $D$, if $C \subseteq A - B$ and $D \subseteq B - A$ then $C$ and $D$ are disjoint.

Let $A$, $B$, $C$, and $D$ be sets such that $C \subseteq A - B$ and $D \subseteq B - A$. Suppose $C$ and $D$ are not disjoint. Then, by definition of disjoint, there exists $x \in C \cap D$, so $x \in C$ and $x \in D$. Now, since $C \subseteq A - B$ and $x \in C$, $x \in A - B$. Similarly, $x \in B - A$. Therefore, by definition of set difference, $x \in A$ and $x \notin B$ and $x \in B$ and $x \notin A$, which is a contradiction. Therefore $C$ and $D$ are disjoint.

Problem 13: Prove or disprove: for any sets $A$ and $B$, $\mathcal{P}(A - B) = \mathcal{P}(A) - \mathcal{P}(B)$.

Counterexample: let $A = \{1, 2\}$ and $B = \{2\}$. Then $\mathcal{P}(A - B) = \mathcal{P}(\{1\}) = \{\emptyset, \{1\}\}$ and $\mathcal{P}(A) - \mathcal{P}(B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\} - \{\emptyset, \{2\}\} = \{\{1\}, \{1, 2\}\}$.

Problem 14: A state issues license plates in two formats: either two letters followed by three numbers, or 5 numbers with a letter somewhere between them (but not at the ends). How many different license plates can the state issue?

There are $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10$ plates of the first kind and $10^5 \cdot 4 \cdot 26$ of the second kind (choose digits, then choose position for letter, then choose the letter) for a total of 11076000.

Problem 15: Imagine a game like poker but played with 4 card hands. Which should be a better hand: one pair or “melting pot”, which is four cards all of different non-consecutive ranks and different suits.

There are $13 \cdot \binom{4}{2} \cdot \binom{12}{2} \cdot 4 \cdot 4 = 82368$ one pair hands and fewer than $13 \cdot 12 \cdot 11 \cdot 10 = 17160$ melting pot hands (that total includes some straights). So melting pot should beat one pair.

Problem 16: 50 people were surveyed about what TV shows they watch. 21 reported that they watch The Amazing Race, 25 watch Veronica Mars, and 11 watch Arrested Development. 3 watch AD and TAR, 9 watch TAR and VM, 3 watch VM and 1 watches all 3. Fill in the Venn diagram that shows how many people watch each possible combination of shows.
Problem 17: 12 people are to be seated for a jury. The jury pool consists of 20 people: 7 college students, 10 retirees, and 3 working professionals. How many possible juries are there? How many have an equal number from each group? How many include no retirees? How many include at most 3 retirees? How many include all the college students? How many include more working professionals than college students?

There are \( \binom{20}{12} \) possible juries. None have an equal number from each group since there aren’t enough working professionals. None include no retirees because there aren’t enough non-retirees. \( \binom{10}{2} + \binom{10}{3} \cdot \binom{10}{9} \) have at most 3 retirees. \( \binom{13}{3} \) include all the college students. \( \binom{7}{2} \cdot \binom{10}{7} \cdot \binom{10}{9} \) include 3 working professionals and 2 college students. \( \binom{7}{1} \cdot \binom{10}{8} \) include 3 professionals and 1 college student, \( \binom{10}{9} \) include 3 professionals and no college student, \( \binom{7}{2} \cdot \binom{10}{1} \cdot \binom{9}{9} \) include 2 professionals and 1 college student, and \( \binom{3}{2} \) includes 2 professionals and no college students, for a total of

\[
\binom{3}{2} + \binom{7}{2} \cdot \binom{10}{7} + \binom{7}{1} \cdot \binom{10}{8} + \binom{10}{9} + \binom{3}{2} \cdot \binom{7}{1} \cdot \binom{10}{9} + \binom{3}{2}
\]

with more professionals than college students.

Problem 18: The game Can’t Stop is played with 4 indistinguishable 6-sided dice. How many distinct outcomes are there of rolling the dice?

\( \binom{9}{4} \).

Problem 19: Define \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = (x - 1)(x + 2)(x + 3) \). Is \( f \) 1-1? Is \( f \) onto? Explain your answers.

\( f \) is not one-to-one because \( f(1) = f(-2) = f(-3) = 0 \). \( f \) is onto: \( \lim_{x \to \infty} = \infty \) and \( \lim_{x \to -\infty} = -\infty \), and \( f \) is continuous, so, by the intermediate value theorem for any \( y \) there is an \( x \) such that \( f(x) = y \).

(Intuitively: \( f \) is a cubic and so goes to positive and negative infinity, so it intersects every horizontal line at least once.)
**Problem 20:** Prove that the set of squares of integers \( \{0, 1, 4, \ldots \} \) is countably infinite by finding a bijection between that set and \( \mathbb{Z}^+ \).

Define \( f : \{0, 1, 4, \ldots \} \rightarrow \mathbb{Z}^+ \) by \( f(x) = \sqrt{x} + 1 \). With \( \{0, 1, 4, \ldots \} \) as the domain and \( \mathbb{Z}^+ \) as the co-domain, \( f \) is one-to-one and onto.

**Problem 21:** Prove that \( \mathbb{Z} \times \mathbb{Z} \) is countably infinite.

Since \( \mathbb{Z} \) is countably infinite, there is a bijection from \( \mathbb{Z} \) to \( \mathbb{N} \). Since \( \mathbb{N} \times \mathbb{N} \) is countably infinite, there is a bijection \( g : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{Z}^+ \). Use \( f \) and \( g \) to define a new function \( h : \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}^+ \) where \( h(a, b) = g(f(a), f(b)) \). Since \( f \) and \( g \) are bijections, \( h \) is too since it is the composition of two bijections (with a Cartesian product thrown in, but it can be shown that that does not affect the one-to-one-ness or the onto-ness).

This is a formalism of the same argument used to show that \( \mathbb{N} \times \mathbb{N} \) is countable: the elements of \( \mathbb{Z} \times \mathbb{Z} \) are arranged in a table with the rows and columns ordered based on \( f \) instead of in simple increasing order (since that would never get to negative numbers).

\[
\begin{array}{cccc}
(f(1), f(1)) & (f(1), f(2)) & (f(1), f(3)) & \ldots \\
(f(2), f(1)) & (f(2), f(2)) & (f(2), f(3)) & \ldots \\
(f(3), f(1)) & (f(3), f(2)) & (f(3), f(3)) & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\end{array}
\]

The elements of the table are then listed diagonal by diagonal just as for the elements of \( \mathbb{N} \times \mathbb{N} \).

**Problem 22:** Is the set of infinite sequences of positive integers countable or uncountable? Justify your answer.

Let \( S = \{s \mid s \text{ is a sequence of positive integers}\} \). \( S \) is uncountable.

Suppose \( f : \mathbb{Z}^+ \rightarrow S \). We will show that \( f \) is not onto and hence is not a bijection by constructing a sequence \( s \) such that \( f(x) \neq s \) for any \( x \in \mathbb{Z}^+ \): let \( s_i = 1^+ \) the \( i \)th element of the sequence \( f(i) \). For any \( x \in \mathbb{Z}^+ \), \( f(x) \neq s \) since they differ in the \( x \)th place. Therefore \( f \) is not onto and hence is not a bijection.

Since \( f \) was arbitrarily chosen, no function from \( \mathbb{Z}^+ \) to \( S \) can be a bijection, so \( S \) is not countably infinite. Since it is clearly not finite, it must be uncountable.

**Problem 23:** Order the following functions so that if \( f \) comes before \( g \), then \( f \) is of order at most \( g \) (\( f(x) \) is \( O(g(x)) \)). Indicate which functions can be switched in the lists (in other words, indicate which are of the same order as each other).

\[
x \log x \quad x \quad 2^x \quad 4^x \quad 8x + 1 \quad x^2 + x + 400 \quad x(\log x)^2
\]

(This won’t be on the exam).
\[ x, \ 8x + 1 \]
\[ x \log x \]
\[ x(\log x)^2 \]
\[ x^2 + x + 400 \]
\[ 2^x \]
\[ 4^x \]