90 + 1 = 91
80-89: 3
70-79: 2
<70: 2

\[ \text{median} = 80, \quad \text{mean} = 76 \frac{3}{11} \]

7) \( a_1 = 6 \), \( a_n = 2a_{n-1} - 4 \) for \( n \geq 2 \)

\[ \forall n \in \mathbb{Z}, \ n \geq 1 \rightarrow a_n \geq 4n \]

Proof: Base cases (\( n = 1 \)): \( a_1 = 6 \)
\[ 4 \cdot 1 = 4, \quad 6 \geq 4 \]
so \( a_1 \geq 4 \cdot 1 \)

\((n=2): a_2 = 8 \)
\[ 4 \cdot 2 = 8, \quad 8 \geq 4 \cdot 2 \]
so \( a_2 \geq 4 \cdot 2 \)

Inductive step: Suppose \( k \geq 2 \) and \( a_k \geq 4k \).

[want \( a_{k+1} \geq 4(k+1) \)]

Then \( a_{k+1} = 2 \cdot a_k - 4 \) (def. of seq)

need \( 8k - 4 \geq 4k + 4 \)
\[ 4k \geq 8 \]
\[ k \geq 2 \]
(\( \text{since } k \geq 2 \))

\[ P(A \land B) = P(A) \cap P(B) \]

Proof: Let \( X \in P(A \land B) \)
\[ Y \in P(\mathbb{Z}) \]

Then \( X \subseteq A \land B \).
\[ \Rightarrow A \land B \subseteq A \text{ and } A \land B \subseteq B \]

So \( X \subseteq A \) and \( X \subseteq B \) (transitivity of \( \subseteq \) and \( \supseteq \))

\[ \therefore X \in P(A) \land P(B) \]

\[ \therefore X \in P(A) \land P(B) \] (def. of \( \cap \))

\[ \therefore (P(A \land B) \subseteq P(A) \land P(B)) \]

Let \( X \in P(A) \land P(B) \)

Then \( X \subseteq P(A) \) and \( X \subseteq P(B) \) (def. of \( \land \))

and \( X \subseteq A \land X \subseteq B \) (def. of \( P \))

Let \( x \in X \).

Then \( x \in A \land x \in B \) (def. of \( \subseteq \))

So \( x \in A \land B \)

\[ \therefore x \in P(A \land B) \]

\[ \therefore x \in P(A \land B) \] (def. of \( P \))
14) For any sets $A$, $B$, $A \subseteq B \implies B^c \subseteq A^c$.

**Proof:** Suppose there are sets $A$, $B$ s.t. $A \subseteq B$ but $B^c \not\subseteq A^c$. Then there is an $x \in B^c$ s.t. $x \notin A^c$. Then $x \notin A$ and hence $x \notin B$ since $A \subseteq B$. \(\Rightarrow\)

\[ A \subseteq B \iff \forall x, x \in A \implies x \in B \quad \text{(def \(\subseteq\))} \]
\[ \implies \forall x, x \notin B \implies x \notin A \quad \text{(contrapos)} \]
\[ \implies \forall x, x \in B^c \implies x \in A^c \quad \text{(def \(\subseteq\))} \]
\[ \implies B^c \subseteq A^c \quad \text{(def \(\subseteq\))} \]

---

random: can't predict what will happen

sample space: set of all possible outcomes

event: subset of sample space

If $S$ is a finite sample space with *equally likely outcomes*, and $E \subseteq S$, then

\[
P(E) = \frac{\# \text{ outcomes in } E}{\# \text{ outcomes in } S} = \frac{\text{N}(E)}{\text{N}(S)}
\]

Ex: flipping 2 coins $S = \{0 \text{ heads}, 1 \text{ head}, 2 \text{ heads}\}$

\[
P(1 \text{ head}) = \frac{\text{N}(\text{1 head})}{\text{N}(S)} = \frac{1}{3}
\]

\[S = \{TT, TH, HT, HH\} \text{ equally likely}\]

1 head = \{TH, HT\} \quad \text{P(1 head) = \frac{\text{N}(\text{TH, HT})}{\text{N}(S)}} = \frac{2}{4} = \frac{1}{2}

Ex: Roll two dice: \( S = \{11, 1, 3, \ldots, 16, 18, \ldots, 26, 31, \ldots\} \)

\[ P(\text{total is 7}) = \frac{P(\{16, 25, 34, 43, 52, 61\})}{N(S)} = \frac{6}{36} = \frac{1}{6} \]

\[ P(\text{total is 6}) = \frac{P(\{11, 12, 13, 14, 15, 21, 22, 23, 24, 31, 32, 33, 41, 42, 51\})}{N(S)} = \frac{15}{26} \]

Ex: \( P(\text{draw a black face card}) = \frac{\# \text{black face cards}}{\# \text{cards total}} = \frac{6}{52} \)

\( n, n+1, \ldots, m \) has \( m-(n-1) = m-n+1 \) entries

1, 2, 3, \ldots, m

100, 105, 110, 115, \ldots, 995

20, 21.5, 22.5, \ldots, 199.5

199.5 - 20 + 1 = 180 entries

Multiplication rule: if an operation has \( k \) steps with

\[ n_1 \text{ outcomes for 1st step} \]

\[ n_2 \text{ outcomes for 2nd step} \]

\[ \vdots \]

\[ n_k \text{ outcomes for kth step} \]

Then \( n_1 \cdot n_2 \cdot \ldots \cdot n_k \) outcomes for op as a whole

Picking a black face card: 1) pick rank 3

2) pick suit 2

\[ 3 \cdot 2 = 6 \]

MD auto license plates: 1) pick 1st letter 26

2) pick 2nd letter 26

3) pick 3rd letter 26

4) pick 1st digit 10

5) pick 2nd digit 10

6) pick 3rd digit 10

\[ 26^3 \cdot 10^3 = 17.5 \text{ million} \]
\[ A = \{1, 2, 3\} \quad B = \{u, v\} \quad C = \{m, n\} \]
\[ A \times B \times C \text{ has } 3 \cdot 2 \cdot 2 = 12 \text{ elemts} \]
1) pick elt of \( A \) 3
2) " " \( B \) 2
3) " " \( C \) 2
\[ \overline{3 \cdot 2 \cdot 2} \]
\[ \text{PIN w/ no repeated digits} \]
1) pick 1st digit 10
2) " 2nd " 9
3) " 3rd " 8
4) " 4th " 7
\[ 10 \cdot 9 \cdot 8 \cdot 7 = 5040 \]

Matos
Sorhoffs CF must be M or N
Newham
Gibbons must RF
Gibbons
1) choose CF 2
2) choose RF 2
3) choose CF 0 choices here 2 choices can't apply mult rule
1) choose CF 2
2) choose LF 2
3) choose RF 2
2 \cdot 2 \cdot 2 = 8
permutation: ordering of a set of objects

Ex: \(a, b, c\) permutations are \(abc, acb, bac, bca, cab, cba\)

building a permutation:
1) pick 1st of \(n\) items
2) pick 2nd of \(n - 1\) items
3) \(\vdots\)
\(n)\) pick \(n\)th of \(n - 1\) items

\# permutations of \(n\) items = \(n(n-1)\cdots 1 = n!\)

\# permutations of QUIET = 5! = 120

\# of perm of QUIET w/ U V = \# perms of \(\{Q, U, V, I, E, T\}\)
= 4!

\(r\)-permutation: permutation of \(r\)-elt subset

\# \(r\)-permutations of \(n\) elt set = \(P(n, r) = \frac{n!}{(n-r)!}\)

\# 4-digit PWS w/ no rep = \(P(10, 4) = \frac{10!}{6!}\)

= \(10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1\)
\(\frac{5!}{x \cdot 3 \cdot 2 \cdot 1}\)

If \(A_1, \ldots, A_k\) are mutually disjoint then

\(N(A_1 \cup \cdots \cup A_k) = N(A_1) + \cdots + N(A_k)\)

Ex: \# 3 digit ints divisible by 5

\(\{x \in \mathbb{Z} \mid 5 \mid x \land 100 \leq x \leq 999\}\)

\(A_1 = \{x \in \mathbb{Z} \mid 100 \leq x \leq 999 \land x \text{ ends in } 0\}\)
\(A_2 = \{x \in \mathbb{Z} \mid 100 \leq x \leq 999 \land x \text{ ends in } 5\}\)
\(A_3 = \{x \in \mathbb{Z} \mid 100 \leq x \leq 999 \land x \text{ ends in } 0\}\)
\(\text{disjoint} \cup A_1 \cup \cup A_2 \cup A_3\)
\[ N(A) = N(B) + N(C) = 90 + 90 = 180 \]

1) pick 1st 2 digits 10, ..., 99 90
2) write 0 1
   \[
   \begin{array}{c|c}
   90 & 1 \\
   \hline
   \end{array}
   \]

A finite, \( B \subseteq A \rightarrow N(A - B) = N(A) - N(C) \)

For any \( B \), \( N(U - B) = N(U) - N(B) \)

\( B \subseteq C \)

\# 4 digit PINs w/ rep = \# 4 digit PINs - \# 4-digit PINs, no rep
   \[
   = 10000 - 5040
   \]
   \[
   = 4960
   \]

\[ P(A^c) = 1 - P(A) \rightarrow P(S - A) = \frac{N(S - A)}{N(S)} = \frac{N(S) - N(\bar{A})}{N(S)} \]

\[ = \frac{N(S)}{N(S)} - \frac{N(A)}{N(C)} \]

\[ = 1 - P(A) \]

\[ N(A \cup B) = N(A) + N(B) - N(A \cap B) \]

\[ N(A \cup B \cup C) = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C) \]

Ex: \# mults of 3 or 5 in 1...1000

\[ A \{ x \in \mathbb{Z}^+ \mid x \leq 1000 \land \left(3 \mid x \lor 5 \mid x\right) \} \]

\[ = \left\{ x \mid x \leq 1000 \land 3 \mid x \right\} \cup \left\{ x \mid x \leq 1000 \land 5 \mid x \right\} \]

\[ = 3, 6, 9, ..., 999 \]

\[ \text{mults of 15} \]

\[ = 333 + 200 - 66 = 467 \]
50 people polled - like O's, Ravens, Redskins

26 liked O's
4 liked O's and Ravens
1 liked all 3
45 liked ≥ 1
18 liked Redskins
5 liked Redskins + Ravens
16 liked Ravens

\[
N(A\cup B)=N(A)+N(B)-N(A\cap B)
\]
\[
N(A\cup B\cup C)=N(A\cup B)+N(B\cup C)-N(B)
\]

r-combination: is a size-r subset

\# of r-combinations of a set of size \( n = \binom{n}{r} \)

"n choose r" choosing an r-perm of n elts

choose 3-combination of \( \{1, 2, 3, 4, 5\} \)

1) choose r elts \( \binom{5}{3} \)

2) permute them \( \frac{r!}{n!} \)

\[
\binom{n}{r} \cdot r! \cdot \frac{n!}{(n-r)!} = \frac{n!}{r!(n-r)!}
\]

1 2 1) choose 1st \( \binom{5}{1} \)
2 1 2) choose 2nd \( \binom{4}{1} \)
3 2 3! 3) choose 3rd \( \binom{3}{1} \)

\[
\{1, 2, 3\} = \{2, 1, 3\}
\]
# ways to rearrange MISSISSIPPI

1) pick pos. for M \( \binom{11}{1} \)
2) pick pos. for 4 I's \( \binom{10}{4} \)
3) pick pos. for 4 S's \( \binom{6}{4} \)
4) pick pos. for 2 P's \( \binom{2}{2} \)

\# rearrangements = \( \binom{11}{1} \binom{10}{4} \binom{6}{4} \binom{2}{2} \)

1) pick pos. for P \( \binom{11}{2} \)
2) " S \( \binom{9}{4} \)
3) " I \( \binom{5}{3} \)
4) " M \( \binom{1}{1} \)

= \( \binom{11}{2} \binom{9}{4} \binom{5}{3} \binom{1}{1} \)

In general \( \binom{n}{n_1} \binom{n-n_1}{n_2} \binom{n-n_1-n_2}{n_3} \ldots \binom{n-n_1-n_2-n_3-\ldots-n_{k-1}}{n_k} \)

12 people, want a 5 person team \# diff teams = \( \binom{12}{5} \)

suppose A, B won't work w/o other
set of teams = teams w/ A and B \cup teams w/ neither

\# " = \( \binom{10}{3} + \binom{10}{5} \)

suppose C, D won't work together

\# teams = \# teams w/ C not D + \# teams w/ D, not C + \# teams w/ neither

= \( \binom{10}{4} + \binom{10}{4} + \binom{10}{5} \)

= \# teams total - \# teams \# with both

= \( \binom{12}{5} - \binom{10}{3} \)
5 men, 7 women

1) pick men \( \binom{5}{3} \)

2) pick women \( \binom{7}{2} \)

\[ \text{\# teams w/3 men and 2 women} = \binom{5}{3} \binom{7}{2} \]

3) choose \# men

4) choose women

\[ \text{\# 5 people teams w/ \leq 1 man} = \text{\# teams w/0 men} + \text{\# w/1 man} \]

\[ = \binom{5}{3} \binom{7}{2} + \binom{5}{4} \binom{7}{1} \]