4) $\mathbb{O}$ has same card as $\mathbb{2Z}$

Need a bijection $f: \mathbb{O} \rightarrow \mathbb{2Z}$

Let $f$ be $\cdots \rightarrow -3, -1, 1, 3, 5, \cdots$

$f(n) = n + 1$ $\cdots \rightarrow -2, 0, 2, 4, 6, \cdots$

$f$ is 1-1 and onto, so $\mathbb{O}$, $\mathbb{2Z}$ have same card.

$f$ is 1-1: Suppose $f(x_1) = f(x_2)$

$x_1 + 1 = x_2 + 1$

$x_1 = x_2$

$f$ is onto: Let $y \in \mathbb{2Z}$. Then $f \left(\frac{y-1}{2}\right) = y$

$y = x + 1$

$x = y - 1$ in $\mathbb{O}$ since $y \in \mathbb{2Z}$ and even - odd is odd

5) $2 \mathbb{Z} = \{ \cdots, -2, 0, 2, \cdots \}$ has same card as $\mathbb{Z} = \{ \cdots, -2, 0, 2, \cdots \}$

Let $f(x) = \frac{2}{25} x$

$f$ is 1-1, onto
12) Let \( W = (a, b) = \{ x \in \mathbb{R} \mid a < x < b \} \)

\( W \) has same card as \((0, 1)\).

[we need 1-1, onto \( f : (0, 1) \to (a, b) \)]

\[ f(0) = a \]
\[ f(1) = b \]

Let \( f(x) = (b-a)x + a \)

\( (0, b-a) \quad m \cdot 1 + a = b \)
\( (a, b) \quad m = b-a \)

\( f \) is 1-1, onto \( (\text{just like other linear fxns}) \)

\( f \) is onto; Let \( y \in (a, b) \)

[want \( x \) s.t. \( (b-a)x + a = y \)]

\( (b-a)x = y - a \)
\( x = \frac{y-a}{b-a} \)

also need \( x \in (0, 1) \)

since \( a < y < b \),

\( 0 < y-a < b-a \)

and \( 0 < \frac{(y-a)}{b-a} < 1 \)

\( x \)

so \( x \in (0, 1) \)
17) \( \mathbb{Q} \) is dense 
\( (\forall r_1, r_2 \in \mathbb{Q}, r_1 < r_2 \rightarrow \exists x \in \mathbb{Q} \text{ s.t. } r_1 < x < r_2) \)

Let \( r_1, r_2 \in \mathbb{Q} \). Then
\[
\frac{r_1 + r_2}{2} \in \mathbb{Q}
\]
and
\[
r_1 < \frac{r_1 + r_2}{2} < r_2
\]

29) \( \overline{\mathbb{Q}} \) is uncountable

Use: \( A, B \) countable \( \rightarrow A \cup B \) is countable

Contrapositive: \( A \cup B \) uncountable \( \rightarrow A \) uncountable \( \lor B \) uncountable

\( \overline{\mathbb{Q}} \cup \mathbb{Q} = \mathbb{R} \)

\( \mathbb{R} \) is uncountable

\( \therefore \overline{\mathbb{Q}} \) or \( \mathbb{Q} \) are uncountable
\( \mathbb{Q} \) are countable

\( \therefore \mathbb{Q} \) are uncountable
\( \mathbb{Q} \) is countable

\[ \mathbb{N} \times \mathbb{N} = \{ (n, 0) \mid n \in \mathbb{N} \} \cup \{ (n, 1) \mid n \in \mathbb{N} \} \cup \{ (n, 2) \mid n \in \mathbb{N} \} \cup \cdots \]

countable union of countable sets

\[ \bigcup_{m=0}^{\infty} \{ (n, m) \mid n \in \mathbb{N} \} \]

Any countable union of countable sets is countable: Let \( A_1, A_2, \ldots \) all disjoint be countable

\( \{ a_{11}, a_{12}, a_{13}, \ldots \} \)

\( \{ a_{21}, a_{22}, a_{23}, \ldots \} \)

\( \vdots \)

List of elts in \( \bigcup \): \( a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, \ldots \)
\[ Q = \left\{ \frac{a}{1} \mid a \in \mathbb{N} \right\} \cup \left\{ \frac{a}{2} \mid a \in \text{odd } \mathbb{N} \right\} \cup \left\{ \frac{a}{3} \mid a \in \mathbb{N} \land a \neq 0 \pmod{3} \right\} \]

Countably many sets in \( U \):
\[
\begin{array}{cccc}
\frac{0}{1} & 1 & \frac{1}{1} & \frac{2}{1} & \frac{3}{1} \\
\frac{1}{2} & \frac{3}{2} & \frac{5}{2} & \frac{7}{2} & \frac{9}{2} \\
\frac{-1}{2} & \frac{-3}{2} & \frac{-5}{2} & \frac{-7}{2} & \frac{-9}{2} \\
\frac{1}{3} & \frac{2}{3} & \frac{4}{3} & \frac{5}{3} & \frac{6}{3} \\
\end{array}
\]

15) \( \{0, 1\}^* \) (set of strings of 0s, 1s) is countably infinite

Proof: List of elts of \( \{0, 1\}^* \):
\[
0, 1, 10, 11, 00, 01, 10, 11, 000, \ldots
\]

Let \( f(w) = \text{pos of } w \text{ on list} \)

1) pick 0 or 1 for 1st = \# strings before \( w \)
2) pick 0 or 1 for 2nd = \# strings shorter than \( w \) +
3) pick 0 or 1 for 3rd = \# strings same length but alphabetically before \( w \)
\[ \sum_{T \in P} \# \text{ strings of length } l \leq l \sum_{l=0}^{l-1} 2^l + \text{ the num whose binary rep w is sum over lengths } \leq l \text{ shorter than w} \sum_{l=0}^{l-1} 2^l + \sum_{i=0}^{l-1} w_i \cdot 2^i \]

\[ f \text{ is 1-1 and onto} \]
\[ \therefore \{0, 1\}^* \text{ and } \mathbb{N} \text{ have same card} \]
\[ \therefore \{0, 1\}^* \text{ are countably infinite} \]

\[ \{0, 1\}^* = \bigcup_{l=0}^{\infty} \{w \in \{0, 1\}^* \mid 1\text{w} = l\} \]
\[ \text{Countable union of countable sets} \]
\[ \therefore \text{ countable} \]

Let \( S = \{ f \mid f: \mathbb{Z} \rightarrow \mathbb{Z} \} \)
\[ = \{ x^2, x+1, \lceil \log_2 (1x+1) \rceil, \{0 \text{ if x odd} \}
\]
\[ \text{...} \]
\[ \{1 \text{ if x even} \} \]

\( S \) is uncountable.

**Proof: Suppose** \( f: \mathbb{Z}^+ \rightarrow S \)

[will show \( f \) is not onto]

Consider \( f(1) \), \( f(2) \), \( f(3) \), \( f(4) \), \( f(5) \), \( f(6) \), ...

\( f(2) \), \( f(3) \), \( f(4) \), \( f(5) \), \( f(6) \), ...

\( f(3) \), \( f(4) \), \( f(5) \), \( f(6) \), ...

...
Construct \( g : \mathbb{Z} \to \mathbb{Z} \) s.t. \( g \neq f(n) \) for all \( n \in \mathbb{Z}^+ \)

Define \( g \) by \( g(n) = \begin{cases} f(n+1)(n) + 1 & \text{if } n \geq 0 \\ 0 & \text{otherwise} \end{cases} \)

Want \( g \neq f(1) \) now \( g \neq f(n) \) for all \( n \)

need \( g(k) \neq f(1)(k) \) since \( g(n-1) \neq f(n)(n-1) \) for some \( k \)

Want \( g \neq f(2) \) \( g(0) = f(1)(0) + 1 \)

need \( g(k) \neq f(2)(k) \) for some \( k \)

\( g(1) = f(2)(1) + 1 \)

set of fns is uncountable

set of computer programs is countable

\( \Rightarrow \) programs are strings of bits

\( \therefore \) some fns have no corresponding program that computes them

Ex: halting problem:
given program and an input to it, determine if prog halts on input
\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \]

\[ \sum_{i=1}^{n} c \cdot a_i = c \sum_{i=1}^{n} a_i \]

\[ \sum_{i=1}^{n} a_i + b_i = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i \]

**Proof:** Base case: \( n=1 \)

\[ \sum_{i=1}^{1} i = 1 \]

\[ 1 \left( \frac{1 \cdot (1+1)}{2} \right) = 1 \]

**Ind step:** Assume \( k \geq 1 \) and

\[ \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \]

\[ \left[ \text{want} \right] \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} \]

Now

\[ \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \]

\[ = \frac{k(k+1)}{2} + (k+1) \]

\[ = \frac{k(k+1) + 2(k+1)}{2} \]

\[ = \frac{(k+1)(k+2)}{2} \]
a) \( S(Eastman, Giant) \ \Rightarrow T \)

b) universal conditional

\[ \forall x, \text{ something true } \Rightarrow \text{ whatever for subset you're saying about that subset} \]

\[ \forall x \in P, L(x, Columbia) \Rightarrow S(x, Giant) \]

\[ \sim (\forall x, P(x)) \]

c) Everyone doesn't shop at 2 diff stores.

\[ \forall x \in P, \sim (\exists y_1, y_2 \in S \text{ s.t. } y_1 \neq y_2 \land S(x, y_1) \land S(x, y_2)) \]

x doesn't shop at 2 diff stores

\[ \exists y \in S \text{ s.t. } S(x, y) \land \forall z \in S \ y \neq z \Rightarrow \sim S(x, y) \]

d) \[ \exists x \in S \text{ s.t. } \forall y \in P \ S(y, x) \Rightarrow L(y, Columbia) \]

all patrons of x live in Col.
\[ A = \{1, 2, 3\} \quad B = \{1\} \]
\[ \mathcal{P}(A - B) = \mathcal{P}(\{2, 3\}) = \{\emptyset, \{2, 3\}\} \]
\[ \mathcal{P}(A) = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\} \]
\[ \mathcal{P}(B) = \{\emptyset, \{1\}\} \]
\[ \mathcal{P}(A) - \mathcal{P}(B) = \{\{2, 3\}, \{1, 2, 3\}\} \]

\[ f(x) = (x-1)(x+2)(x+3) \]
\[ f(1) = 0 = f(-3) \quad \therefore \text{not 1-1} \]

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