Lecture 17: Sizes of Sets

Q: When are sets of the same "size?"

A: If the sets are finite, when they have the same

Is { 3, 1, 9} the same size as {-1, -2, 0}?

How about infinite sets? Is **Z** the same "size" as $\mathbf{Z}^+$?

Definitions:

A set is **countable** if it is finite or it is countably infinite.

A set is **countably infinite** if it can be placed in 1-1 correspondence with $\mathbb{Z}^+$.

Are these sets countable?

{ 2, 4, 8, 16}

$\mathbb{Z}^+$

$\mathbb{Z}$

$\mathbb{Q}^+$

$\mathbb{R}$

Is the set all **real** numbers between 0 and 1countable?

Suppose they are. Then there is a **list** of these numbers that can be put in 1-1 correspondence with Z^+.

Here is the list:

$0 . a_{11}\, a_{12}\, a_{13}\, a_{14} \ldots a_{1n} \ldots$
$0 . a_{21}\, a_{22}\, a_{23}\, a_{24} \ldots a_{2n} \ldots$
$0 . a_{31}\, a_{32}\, a_{33}\, a_{34} \ldots a_{3n} \ldots$
$0 . a_{41}\, a_{42}\, a_{43}\, a_{44} \ldots a_{4n} \ldots$
.
.
$0 . a_{n1}\, a_{n2}\, a_{n3}\, a_{n4} \ldots a_{nn} \ldots$
.
.

Look at the "diagonal elements" $a_{11}a_{22}a_{33}\ldots a_{nn}\ldots$ of the numbers on the list.

Create a number that differs from every number on the list.

It will differ at the very least in the diagonal element.

Let the new number Diff be a decimal and let

Diff $= 0.d_1d_2d_3\ldots d_n\ldots$
where $d_i = 0$ if $a_{ii} \neq 0$ and
$d_i = 1$ if $a_{ii} = 0$.
(remember a_{ii} is the "diagonal element" of the ith number)

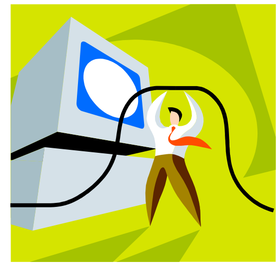
Is Diff *on my list?*

But I claimed that the List contained all real numbers between 0 and 1.

So what does this tell us about the list of all real numbers between 0 and 1?

The size of sets discussion will help us answer the question:

"Can computers do everything?"



If so, then a computer can compute values of any function.

Q: Can a program be written to compute the value of any function I can imagine?

Easier question: Can a program be written to compute the value of any function
$F: \mathbb{Z}^+ \rightarrow \{0, 1, 2, . . . , 9\}$ that I can imagine?

Question: Is the set of all Java programs countable?

Remember *a program is just a string of characters*:

```
public class myStuff
{
 public static void main()
 {
   System.out.println ("Hello world");
 }
}
```

For every real number **r** between 0 and 1, define a function F_r based on that number

$F_r : \mathbb{Z}^+ \rightarrow \{0, 1, 2, \ldots, 9\}$ that "isolates" each decimal place of r. Here's the definition:

if $r = 0 . b_1 b_2 b_3 b_4 \ldots b_n \ldots$, $F_r(1) = b_1$ $F_r(2) = b_2$ $F_r(3) = b_3$ $F_r(4) = b_4 \ldots$ $F_r(n) = b_n \ldots$

Ex. If $r = 0.2316500000$ $F_r(1) =$ $F_r(2) =$ $F_r(3) =$ $F_r(4) =$ $F_r(5) =$ $F_r(6) =$

Ex. If $r = 0.71463$

How many distinct functions F like this are there?

How many Java programs are there?

This is a COUNTING ARGUMENT.

Answer to original question: ______________________________________