Lecture 17: Sizes of Sets

Q: When are sets of the same "size?"

A: If the sets are finite, when they have the same

Is \{3, 1, 9\} the same size as \{-1, -2, 0\}? 

How about infinite sets? Is \(\mathbb{Z}\) the same "size" as \(\mathbb{Z}^+\)?

Definitions:

A set is countable if it is finite or it is countably infinite.

A set is countably infinite if it can be placed in 1-1 correspondence with \(\mathbb{Z}^+\).

Are these sets countable?

\[
\begin{align*}
\{2, 4, 8, 16\} & \quad \mathbb{Q}^+ \\
\mathbb{Z}^+ & \\
\mathbb{Z} & \\
\mathbb{R} & 
\end{align*}
\]

Is the set all real numbers between 0 and 1 countable?
Suppose they are. Then there is a list of these numbers that can be put in 1-1 correspondence with \( Z^+ \).

Here is the list:

\[
\begin{array}{cccccc}
0 & . & a_{11} & a_{12} & a_{13} & a_{14} & \ldots & a_{1n} & \ldots \\
0 & . & a_{21} & a_{22} & a_{23} & a_{24} & \ldots & a_{2n} & \ldots \\
0 & . & a_{31} & a_{32} & a_{33} & a_{34} & \ldots & a_{3n} & \ldots \\
0 & . & a_{41} & a_{42} & a_{43} & a_{44} & \ldots & a_{4n} & \ldots \\
. & . & . & . & . & . & \ldots & . & . \\
0 & . & a_{n1} & a_{n2} & a_{n3} & a_{n4} & \ldots & a_{nn} & \ldots \\
. & . & . & . & . & . & \ldots & . & . \\
\end{array}
\]

Look at the “diagonal elements” \( a_{11} a_{22} a_{33} \ldots a_{nn} \ldots \) of the numbers on the list.

Create a number that differs from every number on the list.

It will differ at the very least in the diagonal element.

Let the new number \( \text{Diff} \) be a decimal and let

\[
\text{Diff} = 0.d_1 d_2 d_3 \ldots d_n \ldots
\]

where \( d_i = 0 \) if \( a_{ii} \neq 0 \) and \( d_i = 1 \) if \( a_{ii} = 0 \).

(remember \( a_{ii} \) is the "diagonal element" of the ith number)

\( \text{Is Diff on my list?} \)

But I claimed that the List contained all real numbers between 0 and 1.

So what does this tell us about the list of all real numbers between 0 and 1?

The size of sets discussion will help us answer the question:
"Can computers do everything?"

If so, then a computer can compute values of any function.

**Q:** Can a program be written to compute the value of any function I can imagine?

**Easier question:** Can a program be written to compute the value of any function 
$$F: \mathbb{Z}^+ \rightarrow \{0, 1, 2, \ldots, 9\}$$ that I can imagine?

**Question:** Is the set of all Java programs countable?

Remember a program is just a string of characters:

```java
public class myStuff {
    public static void main() {
        System.out.println("Hello world");
    }
}
```
For every real number $r$ between 0 and 1, define a function $F_r$ based on that number $r$:

$$F_r : \mathbb{Z}^+ \rightarrow \{0, 1, 2, \ldots, 9\}$$

that “isolates” each decimal place of $r$. Here’s the definition:

if $r = 0.b_1b_2b_3 \ldots b_n \ldots$,

then $F_r(1) = b_1$  \hspace{1cm} $F_r(2) = b_2$  \hspace{1cm} $F_r(3) = b_3$  \hspace{1cm} $F_r(4) = b_4$  \hspace{1cm} $F_r(n) = b_n$  \hspace{1cm} $\ldots$  

Ex. If $r = 0.2316500000$  \hspace{1cm} $F_r(1) = \hspace{1cm} F_r(2) = \hspace{1cm} F_r(3) = \hspace{1cm} F_r(4) = \hspace{1cm} F_r(5) = \hspace{1cm} F_r(6) = \hspace{1cm}$

Ex. If $r = 0.71463$

How many distinct functions $F$ like this are there?

How many Java programs are there?

\textit{This is a COUNTING ARGUMENT.}

Answer to original question: _____________________________________________