Lecture 24: Proving a Function is O(g(x))

Recall the definition of big-O:

f(x) is **O(g(x))** if there exists an x value $\mathbf{x_0}$ and some constant **M** such that
$| f(x) | \leq \mathbf{M} \cdot | g(x) |$ for all $x > x_0$.

Prove $f(x) = 3x^2 + 5x + 8$ is $O(x^2)$

We need a "stretching" factor, M and a x_0 so that
$| f(x) | \leq M \cdot | g(x) |$ for all $x > x_0$.

$| f(x) | = |3x^2 + 5x + 8|$

1. Use Δ Inequality to "break up" the terms
2. Remove | | and list conditions.
3. Substitute lower powers with the higher powers.
   List conditions. ($x > a$, $x > b$, ….)
4. Combine terms to find M
5. Choose the largest value (steps 2 & 3) as x_0

Is f(x) also $O(x^3)$?

$O(x^4)$?

$O(x)$?

ex. Show $f(x) = 3x^5 + 5x^3 - 4x$ is $O(x^5)$.

Polynomial functions are O(x^h) where h is the highest power in the polynomial.
(The highest power term "dominates" all other terms.)

ex. Show $f(x) = | 2x^2 - 5x - 3 |$ is $O(x^2)$.

ex. Show $f(x) = 5 \cdot 2^x + 3\, x^4$ is $O(2^x)$.

Big-O can be a "loose" bound. We would like the ***best*** *Big-O—answer THETA.*

Positive function f(x) is $\Theta(g(x))$ if there exists an x value x_0 and some constants M and N such that $N \cdot g(x) \leq f(x) \leq M \cdot g(x)$ for all $x > x_0$.

Intuition:

BTW, there is a "partner to big-O. It is **Ω**. A function f is Ω(g(x)) if ∃ M and x_0 ∋
$M \cdot g(x) \le f(x) \quad \forall \; x > x_0$.

Intuition:

So if f(x) is both O(g(x) and Ω(g(x)), then it is

<u>To prove f(x) is Θ (g(x)),</u> show

1. f(x) is O(g(x)) (i.e., find "stretching" factor M and x_0 such that $f(x) \le M \cdot g(x)$ for all $x > x_0$.)
2. Find the lower "stretching" factor N and x_0 such that $N \cdot g(x) \le f(x)$ for all $x > x_0$. (i.e. show that f(x) is ____________)
3. Choose the larger of the 2 x_0 as the final x_0. Rewrite the triple inequality, replacing N and M.

Show $f(x) = 7x + 3$ is $\Theta(x)$

Step 1: (big-O)

Step 2:

1. Remove | | and list the conditions.
2. Make quantity smaller by substituting or omitting terms. List conditions.
3. Combine terms to find N
4. Choose the largest value (steps 1 & 2) as x_0

Step 3:

Show $f(x) = 2x^2 - 3x$ is $\Theta(x^2)$.

Show $f(x) = | 3x^2 - 5x + 4|$ is $O(x^2)$.

Show $f(x) = 3 \cdot 2^x + 3 x^3$ is $\Theta(2^x)$.

When a computer scientist says an algorithm is big-O of something, he/she usually means it is Θ of something (tight big-O).