Lecture 12: Graph Theory

Graphs in CS?

Terminology:

Vertices

Edges

Adjacent vertices

Loop

Edge is *incident on* endpoints

More terminology:

**Simple graph**

- no loops
- no more than 1 edge between 2 given vertices

**Complete graph** – all pairs of vertices are connected by exactly 1 edge

Draw a complete draw with 5 points

Degree of a vertex

In C what is the degree of each vertex in the set with m points?
In C what is the degree of each vertex in the set with n points?
Total degree of the graph (Handshake Theorem)

Total degree of graph G = 2 \cdot (\text{number of edges in G})

The number of vertices of odd degree in any graph must be even. Why?

ex. Construct a graph with 4 vertices of degree 1, 2, 2, and 4.

ex. Construct a simple graph with 4 vertices of degree 2, 2, 3, and 5.

Is it possible to construct a graph with any given number of vertices with given degrees?

ex. 6 vertices with degrees 1, 1, 2, 4, 6, 6?

ex. A simple graph with 4 vertices of degrees 2, 2, 4, and 4?

What is the total degree of a complete graph with n vertices?
For graphs, we are interested in certain sequences of vertices and edges (trips around the graph).

A walk from \( v_i \) to \( v_j \) is a sequence of vertex-edge-vertex (where the edge is incident on both vertices) and the first vertex is \( v_i \) and the last vertex is \( v_j \).

A path from \( v \) to \( w \) is a walk with no repeated edges.

A simple path from \( v \) to \( w \) is a path that has no repeated vertex.

A circuit is a closed walk with no repeated edges (repeated vertices OK).
   (Another way to say this is to say that it is a path from a vertex to itself.)

A simple circuit is a circuit with no repeated vertex—except the first and last. (a simple path from a vertex to itself)

An Euler circuit of a graph is one that involves every edge and every vertex of the graph (i.e., every edge is traversed exactly once).
The Bridges of Konigsberg:

Is there a way to start and end at the same point and cross each bridge exactly once?

Is there an Euler circuit of the graph?

Could a circuit be an Euler circuit AND be simple?

A connected graph has an Euler circuit if and only if each of its vertices has an even degree.

Finding an Euler circuit:
Start with a circuit of a subgraph; “patch in” additional circuits of subgraphs until every vertex and edge is included.

Hamiltonian Circuit: a simple circuit in which each vertex is visited
( simple = a vertex may not be visited more than once—unless it's the starting point circuit = no repeated edges; starting pt. = stopping pt. )

To have a Hamiltonian circuit, the graph must have a subgraph that
1. is connected
2. contains every vertex
3. has each vertex of degree 2
4. has number edges = number vertices

Do the graphs below have Hamiltonian circuits?
Euler circuit - all edges traversed once (remember the bridges)

Hamiltonian circuit - all vertices traversed once

Finding an Euler circuit can be done in $O(n)$.

Finding a Hamiltonian circuit?

A very famous problem: Traveling Salesperson: A weighted graph

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Problems that can be solved in Polynomial Time (Tractable Problems or Class P)

Problems that need Exponential Time (Intractable Problems)

Class NP: Problems whose solutions can be verified in polynomial-time

Class NP-complete
A set of "hard" problems.

P = NP?

Famous NP-Complete Problems: