Lecture 9: Recursion
recurrere = to run back (L)

Recursively defined sequence: the k-th term is defined in terms of 1 or more of the preceding terms; one or more initial terms must be given

\[ a_k = 2a_{k-1} + 3 \quad a_0 = 1 \]

\[ a_k = (a_{k-2})^2 \quad a_0 = 2 \quad a_1 = 3 \]

ex. 1 Fibonacci sequence (Rabbits)

<table>
<thead>
<tr>
<th>Month</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
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<tbody>
<tr>
<td># Pairs</td>
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ex. 2 The number of individual disk-moves needed to solve the Towers of Hanoi puzzle

Let \( m_k \) be the total number of individual disk-moves if the puzzle has \( k \) disks.
ex. 3  Compound interest after $k$ years at rate $r$

Let $A_k$ be the total amount after $k$ years.

ex. 4  The number of strings in $\Sigma^k$ where $\Sigma = \{ 0, 1 \}$ that do not contain the pattern “11”

Hint: How can such strings start?

ex. 5  The number of ways of choosing $k$ objects out of a set of $n$ objects $S_{n,k}$

Hint: Focus on 1 point ♦
ex. 6 How many ways can you partition a set with n elements into r subsets? $S_{n,r}$

Easy cases:

$S_{1,1}$

$S_{2,1}$

$S_{2,2}$

$S_{n,1}$

$S_{n,n}$

Hint: Pick a point *. In each partition, either {*} is a set or * is part of another set.

Solving recurrence relations

Solving a recurrence relation means

Methods:
A. Iteration

Idea: Substitute until you see a pattern, then make an intelligent guess. (The guess can be proved by induction.)

ex: The compound interest recurrence relation
ex: the Towers of Hanoi puzzle. Let $T(n)$ be the number of disk-moves to solve the puzzle with $n$ disks.

Object: Move the entire stack of disks from peg A to peg C

Rules: Move 1 disk at a time.
Place no larger disk atop a smaller disk.
You may use peg B as a helper or "auxiliary" peg.

A

source peg

B

destination peg

C

$T(1) =$

$T(2) =$

$T(3) =$
Second-Order Linear Homogeneous Recurrence Relations with Constant Coefficients

Techniques for finding explicit formulas for special cases of recursively defined sequences.

Definition: A second-order linear homogeneous recurrence relation with constant coefficients is a recurrence relation of the form

\[ a_k = Aa_{k-1} + Ba_{k-2} \]

for all integers \( k \geq \) some fixed integer, where \( A \) and \( B \) are fixed real numbers with \( B \neq 0 \).

Determine whether each of the following is a second-order linear homogeneous recurrence relation with constant coefficients:

1. \( a_k = 3a_{k-1} + 2a_{k-2} \)
2. \( b_k = \frac{1}{2}b_{k-1} + \frac{1}{2}b_{k-2} \)
3. \( c_k = c_{k-1} + c_{k-2} + c_{k-3} \)
4. \( d_k = d_{k-1}^2 + d_{k-1} \cdot d_{k-2} \)
5. \( e_k = 2e_{k-2} \)
6. \( f_k = 2f_{k-1} + 1 \)
7. \( g_k = g_{k-1} + g_{k-2} \)
8. \( h_k = (-1)h_{k-1} + (k - 1)h_{k-2} \)

Consider a SOHLrrCC:

\[ a_k = Aa_{k-1} + Ba_{k-2} \]

for all integers \( k \geq 2 \), where \( A \) and \( B \) are fixed real numbers. Suppose that for some number \( t \) with \( t \neq 0 \), the sequence

\[ 1, t, t^2, t^3, \ldots, t^n, \ldots \]

satisfies the relation.

Find \( t^k \).

Derive the characteristic equation:

Use the characteristic equation to find solutions to a Recurrence Relation

Ex. \( a_k = a_{k-1} + 2a_{k-2} \) for all integers \( k \geq 2 \). Find all sequences that satisfy the relation and have the form \( 1, t, t^2, t^3, \ldots, t^n, \ldots \), where \( t \) is nonzero.
Lemma 8.3.2
If \( r_1, r_2, \ldots \) and \( s_1, s_2, s_3, \ldots \) are sequences that satisfy the same second-order linear homogeneous relation with constant coefficients, and if \( C \) and \( D \) are any numbers, than the sequence \( a_0, a_1, a_2, \ldots \) defined by the formula
\[
a_n = Cr_n + Ds_n
\]
for all integers \( n \geq 0 \) also satisfies the same recurrence relation.

**Example 1**: Find a sequence that satisfies the recurrence relation \( a_k = a_{k-1} + 2a_{k-2} \) and also the initial conditions \( a_0 = 1 \) and \( a_1 = 8 \).

**Example 2**: Find a sequence that satisfies the recurrence relation \( a_k = 2a_{k-1} + 3a_{k-2} \) and also the initial conditions \( a_0 = 1 \) and \( a_1 = 2 \).
Theorem 8.3.3 - Distinct-Roots Theorem
Suppose a sequence \( a_0, a_1, a_2, \ldots \) satisfies a recurrence relation
\[
a_k = Aa_{k-1} + Ba_{k-2}
\]
for some real numbers \( A \) and \( B \) with \( B \neq 0 \) and all integers \( k \geq 2 \). If the characteristic equation
\[
t^2 - At - B = 0
\]
has two distinct roots \( r \) and \( s \), then \( a_0, a_1, a_2, \ldots \) satisfies the explicit formula
\[
a_n = Cr^n +Ds^n,
\]
where \( C \) and \( D \) are the numbers whose values are determined by the value \( a_0 \) and \( a_1 \).

A Formula for the Fibonacci Sequence